

# Geometric Structure of Two Self-dual Fields with Constraints

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## Abstract

A two dimensional Poincaré-invariant self-dual field with constraints is studied in geometric way. We obtained its symplectic structure and conservative currents on space of solutions, which are also invariant under transformations of Poincaré group.

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## INTRODUCTION

Symplectic manifold associated with a time-dependent Lagrangian in classical mechanics is the space of motions. The correspondence in field theory is the infinite-dimensional symplectic manifold of classical solutions of the field equations. Unfortunately, it gives a particular canonical coordinate system which lets violate covariance because it selects out a particular coordinate system. It is not inherent in the canonical approach. In geometric approach, the coordinates are irrelevant since all we just need symplectic structure [1]. The symplectic structure can be obtained directly from the Lagrangian without violating covariance.

The geometric approach is a powerful tool to study classical mechanics systems and their symmetry. For a given classical field, space of motions is its phase space [2]. Symplectic structures on space of motions can be obtained from its Lagrangian. From Poincaré invariant symplectic structure is, the classical field's conservative currents can be calculated. Hence we can give out geometric descriptions of classical fields.

In this paper, we will apply the above approach to self-dual field. self-dual fields  $\phi(x, t)$ , sometimes called chiral bosons, satisfy the self-dual condition,

$$\dot{\phi} = \phi', \quad (1)$$

where the overdot denotes differentiation with respect to time  $t$  and prime to space  $x$ . The self-dual fields have been studied widely. Yang and Luo [3] discussed geometric structure for free two-dimensional self-dual field. In this paper self-dual field with constraints is studied. By symplectic technique, we obtained its geometric description.

## HAMILTONIAN DESCRIPTION

In this paper we study the following Lagrangian density with constraints [4]

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 + \lambda(\dot{\phi} - \phi'), \quad (2)$$

where  $\lambda = \lambda(x, t)$  is an auxiliary field. The equations of motions are

$$\dot{\phi} = \phi', \quad \dot{\lambda} = \lambda'. \quad (3)$$

So it is a self-dual field.

Firstly let us study the canonical treatment of this system. To obtain the canonical formalism, we should treat  $\phi$ ,  $\lambda$  and their corresponding conjugate canonical momenta  $\pi$ ,  $p$  at each point  $(t, x)$  as independent coordinates on the momentum phase space  $V$  [4].

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \dot{\phi} + \lambda, \quad (4)$$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{\lambda}} = 0, \quad (5)$$

Eq.( 5) is a primary constraint,

$$\Omega_1 := p \approx 0. \quad (6)$$

The secondary constraint is

$$\Omega_2 := \partial_t \Omega_1 = \pi - \lambda - \phi' \approx 0. \quad (7)$$

With these two constraints, reduction of phase space becomes to have only two degrees of freedom.

If we can choose  $(\pi, \phi)$  as two independent variables, Hamiltonian is

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L} = \frac{1}{2}(\pi - \lambda)^2 + \frac{1}{2}\phi'^2 + \lambda \phi' = \pi \phi'. \quad (8)$$

The Poisson brackets among the constraints are

$$\{\Omega_1(x, t), \Omega_1(y, t)\} = 0, \quad (9)$$

$$\{\Omega_1(x, t), \Omega_2(y, t)\} = \delta(x - y), \quad (10)$$

$$\{\Omega_2(x, t), \Omega_2(y, t)\} = -2\partial_1 \delta(x - y). \quad (11)$$

From these commutation relations, Dirac brackets  $\{, \}_D$  were calculated, and then quantization were discussed in [4].

## SYMPLECTIC STRUCTURE

For any two vectors  $\psi = (\psi^\alpha)$  and  $\bar{\psi} = (\bar{\psi}^\beta)$  on space of solutions  $\mathcal{M}$ , symplectic structure on  $\mathcal{M}$  is [1]

$$\omega(\psi, \bar{\psi}) = \frac{1}{2} \int_{\Sigma} \omega^\nu d\sigma_\nu, \quad (12)$$

where  $d\sigma_\nu = n_\nu d\sigma$ ;  $d\sigma$  is the volume element on  $\Sigma$  and  $n_\nu$  is the unit normal.  $\Sigma \subset Q$  is a spacelike hypersurface in Minkowski space.  $\omega = d\theta$  is independent of  $\Sigma$  and is a natural closed 2-form on  $\mathcal{M}$  [1].

$$\omega^\nu = \frac{\partial^2 \mathcal{L}}{\partial \psi^\beta \partial \psi_\nu^\alpha} (\psi^\beta \bar{\psi}^\alpha - \bar{\psi}^\beta \psi^\alpha) + \frac{\partial^2 \mathcal{L}}{\partial \psi_b^\beta \partial \psi_\nu^\alpha} (\bar{\psi}^\alpha \nabla_b \psi^\beta - \psi^\alpha \nabla_b \bar{\psi}^\beta), \quad \nu = 0, 1. \quad (13)$$

, In our case,  $\psi = (\phi, \lambda)$ ,  $\bar{\psi} = (\bar{\phi}, \bar{\lambda})$ . Its symplectic structure

$$\omega^0 = (\lambda\bar{\phi} - \bar{\lambda}\phi) + (\bar{\phi}\dot{\phi} - \phi\dot{\bar{\phi}}), \quad (14)$$

$$\omega^1 = (\bar{\lambda}\phi - \lambda\bar{\phi}) + (\phi\bar{\phi}' - \bar{\phi}\phi'), \quad (15)$$

we have

$$\partial_\nu \omega^\nu = \partial_t \omega^0 + \partial_x \omega^1 = 0. \quad (16)$$

So  $\omega^\nu$  ( $\nu = 0, 1$ ) are conservative symplectic currents.

Now let us consider two dimensional conformal transformation,

$$\delta_f x_\mu = x'_\mu - x_\mu = f_\mu, \quad (17)$$

$\mu = 0, 1$ , which is larger than the Poincaré group of translations and Lorentz rotations [5]. Conformal Killing vector equation for this transformation is

$$\partial_\mu f_\nu + \partial_\nu f_\mu = \partial_\lambda f^\lambda g_{\mu\nu}. \quad (18)$$

Its Lie algebra

$$[\delta_f, \delta_g] = \delta_{(f,g)}, \quad (19)$$

where  $(f, g)_\mu = f^\alpha \partial_\alpha g_\mu - g^\alpha \partial_\alpha f_\mu$ ,  $\alpha = 0, 1$ . For self-dual field  $\phi(x, t)$  we have

$$\delta_f \phi = f^\mu \partial_\mu \phi. \quad (20)$$

$$\Delta\omega = \int_\Sigma \delta\omega^\nu d\sigma_\nu, \quad (21)$$

$$\Delta\omega^0 = (f^0 + f^1)(\dot{\lambda}\bar{\phi} + \lambda\dot{\bar{\phi}} - \dot{\lambda}\phi - \bar{\lambda}\dot{\phi} + \ddot{\phi}\bar{\phi} - \phi\ddot{\bar{\phi}}); \quad (22)$$

$$\Delta\omega^1 = (f^0 + f^1)(\dot{\bar{\lambda}}\phi + \bar{\lambda}\dot{\phi} - \dot{\bar{\lambda}}\bar{\phi} - \bar{\lambda}\dot{\bar{\phi}} + \phi\dot{\bar{\phi}}' - \dot{\phi}'\bar{\phi}). \quad (23)$$

if we require

$$\Delta\omega = 0, \quad (24)$$

then there must be

$$f^0 + f^1 = 0. \quad (25)$$

It means the two-dimensional conformal group contracts to one component. This result is consistent with [6].

## CONSERVATIVE CURRENTS

Generally, for a given Lagrangian system, according to Noether theorem, we can study its conservative currents with symmetries of the Lagrangian by geometric approach.

let  $Q$  be background space-time, there is a Killing vector  $V(x)$  on it generating a flow  $\rho_t$

$$\rho_t = Q \rightarrow Q. \quad (26)$$

$$\frac{d}{dt}|_{t=0}\rho_t(x) = V(x), \quad x \in Q. \quad (27)$$

$\rho_t$  induces a “push forward” operator  $R_t$

$$R_t(\phi) = \phi \cdot \rho_t^{-1}, \quad (28)$$

If  $\phi$  is a solution of a field equation, then so is  $R_t\phi$ . So  $R_t$  maps solutions to solutions, and hence induces a flow on  $\mathcal{M}[1]$ . We call  $R_t$  is hamiltonian current on the manifold  $\mathcal{M}$ , which preserves Hamilton. Hamiltonian vector field  $X$  is defined by:

$$\frac{d}{dt}(R_t\phi^\alpha)_{t=0} = X^\alpha, \quad (29)$$

and conservative currents are

$$J_\mu = X(\phi) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + V^\mu(x) \mathcal{L}. \quad (30)$$

For the Poincaré-invariant self-dual field, it is easy to obtain its various conservative geometric currents:

- We first consider time and space translations. Conserved quantities are total energy and total momentum:

$$H = \int (\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + \lambda\phi') dx. \quad (31)$$

$$P = -H. \quad (32)$$

- For Lorentz transformations

$$x'_\mu = a_{\mu\nu} x_\nu \quad \mu, \nu = 0, 1, \quad a_{\mu\nu} = \delta_{\mu\nu} + \alpha_{\mu\nu}, \quad (33)$$

$\alpha_{\mu\nu}$  is a infinite asymmetry. Its Killing vector

$$V^\mu(x) = \frac{\partial}{\partial s}|_s \alpha_{\mu\nu}(s) x_\nu, \quad \beta_{\mu\nu} \equiv \frac{\partial}{\partial s}|_s \alpha_{\mu\nu}, \quad (34)$$

and Hamiltonian vector

$$X(\phi) = \frac{d}{ds}\big|_{s=0}\phi'(\rho_s x) = \partial_\lambda \phi \beta_{\lambda\nu} x_\nu. \quad (35)$$

It generates conservative currents

$$J_\mu = \beta_{\lambda\nu} x_\nu \partial_\lambda \phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \beta_{\mu\nu} x_\nu \mathcal{L} = \frac{1}{2} \beta_{\mu\nu} (x_\nu T_{\mu\lambda} - x_\lambda T_{\mu\nu}) = \frac{1}{2} \beta_{\mu\nu} L_{[\lambda\nu]\mu}, \quad (36)$$

where  $L_{[\lambda\nu]\mu}$  is the 3rd orbital angular momentum tensor. The  $J^\mu$  corresponds to total orbital angular momentum.

- Since the self-dual field is invariant under one-parameter two-dimensional conformal transformations (with condition  $f^0 + f^1 = 0$ ), so we also can study conservative currents it generates.

$$x'_\mu = x_\mu + f_\mu(x(s)), \quad (37)$$

$$V^\mu(x) = \partial_\nu f_\mu \frac{d}{ds}\big|_{s=0} x_\nu(s) + \frac{d}{ds}\big|_{s=0} x_\mu(s) = (\delta_{\mu\nu} + \partial_\nu f_\mu) u_\nu, \quad (38)$$

where  $u_\nu \equiv \frac{d}{ds}\big|_{s=0} x_\nu(s)$ .

$$X(\phi) = \frac{d}{ds}\big|_{s=0}\phi'(\rho_s x) = \partial_\lambda \phi u_\lambda, \quad (39)$$

$$J_\mu = \partial_\lambda \phi u_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + (\delta_{\mu\nu} + \partial_\nu f_\mu) u_\nu \mathcal{L} = u_\lambda T_{\lambda\mu} + \partial_\nu f_\mu u_\nu \mathcal{L}. \quad (40)$$

## CONCLUSIONS

We study Poincaré-invariant two dimensional self-dual field with constraints. Symplectic structure and conservative geometric currents on space of motions were obtained. It made the first step for geometric quantization of self-dual fields.

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